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LETTER TO THE EDITOR

Entropic uncertainty relations for the infinite well

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Abstract. We calculate the entropic and standard uncertainty relations for the position and momentum of the infinite potential well as functions of its quantum states. We show that the entropic uncertainty relations express more adequately the uncertainty principle than the standard ones which use the dispersions as measures of the uncertainties for two non-commuting observables.

1. Introduction

In recent years there has been considerable interest in determining the entropies of the non-commuting observables of quantum systems mainly in order to formulate their entropic uncertainty relations [1–3]. An entropic uncertainty relation represents the sum of entropies of two non-commuting observables and is constructed as follows: consider a normalized state vector $|\psi\rangle$ in an n -dimensional Hilbert space and let the observables A and B have non-degenerative spectra of eigenvectors $|a\rangle$ and $|b\rangle$, respectively. The entropic uncertainty relation is an inequality of the form [1]

$$S_A + S_B \geq S_{AB} \quad (1)$$

where

$$S_A = - \sum_i |\langle \psi | a_i \rangle|^2 \log |\langle \psi | a_i \rangle|^2$$

$$S_B = - \sum_j |\langle \psi | b_j \rangle|^2 \log |\langle \psi | b_j \rangle|^2$$

and S_{AB} is a positive constant which represents the lower bound of the right-hand side of the inequality (1). For the continuous observables A_c and B_c described by the wavefunctions $\psi(x)$ and $\phi(p)$, the inequality (1) reads

$$S_{A_c} + S_{B_c} \geq S_{AB}$$

where

$$S_{A_c} = - \int_{-\infty}^{\infty} |\psi(x)|^2 \log |\psi(x)|^2 dx$$

$$S_{B_c} = - \int_{-\infty}^{\infty} |\phi(p)|^2 \log |\phi(p)|^2 dp.$$

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If one takes for the non-commuting observables the position x and momentum p of a quantum system then the corresponding entropic uncertainty relation has the following form

$$S_x + S_p \geq S_{xp} \quad (1a)$$

where

$$S_x = - \int_{-\infty}^{\infty} q(x) \log q(x) dx$$

and

$$S_p = - \int_{-\infty}^{\infty} r(p) \log r(p) dp$$

where $q(x)$ and $q(p)$ are the probability density functions of the position and momentum, respectively.

As is well known, the standard uncertainty relation is usually given in the form of the Robertson formula [10]

$$\Delta A \Delta B \geq \frac{1}{2} |\langle \Psi | [A, B] | \Psi \rangle| \quad (2)$$

where ΔA and ΔB are the dispersions of two non-commuting observables A and B and $[A, B]$ is their commutator. It has been pointed by many authors that the Robertson form of uncertainty relations has two serious shortcomings.

(i) The right-hand side of inequality (2) is not a fixed lower bound, but it depends on the quantum state of the considered quantum system. If the observable A or B is in its eigenstate then $[A, B] = 0$ and no restriction on ΔA or ΔB is imposed by the left-hand side of the inequality (2).

(ii) The dispersion may not represent the appropriate measure for the uncertainty of an observable if its probability distribution exhibits some sharp distant peaks (for a detailed discussion see [13]).

It has been found by many authors that the first-mentioned shortcoming does not occur if one uses Shannon's entropy for the uncertainties of the observables. We next show that in the case of the infinite well its momentum entropy represents a more adequate measure of uncertainty than does its dispersion.

2. Position–momentum uncertainty relations for the potential well

Position–momentum uncertainty relations, both entropic and standard, have been found for many important quantum systems [2, 3, 7–9]. In what follows we determine them for the infinite potential well. This quantum system can nicely illustrate the advantage of expressing the uncertainty of position and momentum by means of their entropies (as in the entropic uncertainty relation) instead of their dispersions (as in the standard uncertainty relation). This is why the momentum probability density function of the potential well exhibits sharp distant peaks for the quantum number $n > 1$. In order to see this we plot the position and momentum probability distribution in figure 1 for $n = 1, 5$ and 10 and note that two sharp distant peaks occur in the momentum probability distribution except for the ground state. It can be readily shown that the distance between the sharp peaks in the probability density are located at $\xi = \pm (n - 1/2) \pi$ (see figure 1).

In our further consideration we take a symmetric well potential defined as

$$V(x) = 0 \text{ for } |x| < a \quad \text{and} \quad V(x) = +\infty \text{ for } |x| > a.$$

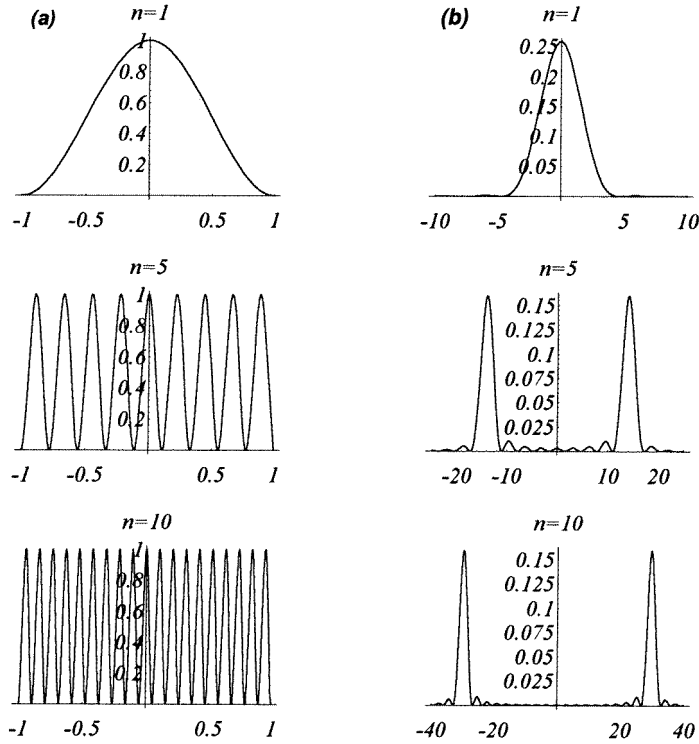


Figure 1. The position probability densities (a) and momentum probability densities (b) for the symmetric infinite well corresponding to $n = 1, 5, 10$ (top, middle, bottom, respectively).

For the sake of simplicity we take only even solutions of the corresponding Schrödinger equation. The symmetric eigenfunctions of the Schrödinger equation are [4, 6]

$$u_n^+ = \frac{1}{\sqrt{a}} \cos \left[\frac{(n-1/2)\pi x}{a} \right] \quad n = 1, 2, 3, \dots$$

with energy eigenvalues

$$E_n^+ = \frac{\hbar^2(2n-1)^2\pi^2}{8ma^2}.$$

The corresponding momentum wavefunctions are given as the Fourier transforms of u_n^+ :

$$\begin{aligned} \phi_n^+(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-a}^{+a} u_n^{(+)}(x) \exp(-ipx/\hbar) dx \\ &= \sqrt{\frac{a}{2\pi\hbar}} \left\{ \frac{\sin[(n-1/2)\pi - ap/\hbar]}{[(n-1/2)\pi - ap/\hbar]} + \frac{\sin[(n-1/2)\pi + ap/\hbar]}{[(n-1/2)\pi + ap/\hbar]} \right\}. \end{aligned}$$

Since u_n^+ and $\phi_n^+(p)$ are symmetric functions the dispersion of position and momentum Δx and Δp as a function of quantum number n is given by the integral

$$(\Delta x(n))^2 = 2 \int_0^a \frac{1}{a} \cos \left[\frac{(n-1/2)\pi x}{a} \right]^2 x^2 dx$$

and

$$(\Delta p(n))^2 = \left(\frac{\hbar}{2\pi a} \right)^2 2 \int_0^\infty \Phi(\xi) \xi^2 d\xi$$

where

$$\Phi(\xi) = \left\{ \frac{\sin[(n-1/2)\pi - \xi]}{[(n-1/2)\pi - \xi]} + \frac{\sin[(n-1/2)\pi + \xi]}{[(n-1/2)\pi + \xi]} \right\}^2$$

and $\xi = ap/\hbar$, respectively.

The entropy of position and momentum is

$$S_x(n) = - \int_{-a}^a \left\{ \cos \left[\frac{(n-1/2)\pi x}{a} \right] \right\}^2 \log \left\{ \cos \left[\frac{(n-1/2)\pi x}{a} \right] \right\}^2 dx$$

and

$$S_p(n) = - \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\xi) \log \Phi(\xi) d\xi$$

respectively.

The dispersions of position and momentum Δx and Δp for the potential well as functions of n are known analytically (see, e.g., [11])

$$\Delta x = \frac{a}{\sqrt{3}} \sqrt{1 - \frac{6}{n^2\pi^2}} \quad (3a)$$

and

$$\Delta p = \frac{\hbar}{2} \left[\frac{(2n-1)\pi}{a} \right]. \quad (3b)$$

Inserting $\Delta x(n)$ and $\Delta p(n)$ into the uncertainty product $\Delta x \Delta p$, we get

$$\Delta x \Delta p = \frac{\hbar}{2} \left[\frac{(2n-1)\pi}{\sqrt{3}} \sqrt{1 - \frac{6}{n^2\pi^2}} \right].$$

It can be easily shown that for large n the position and momentum dispersions tend to $a/\sqrt{3}$ and $(\hbar n\pi)/a$, respectively, and the uncertainty product becomes a linear function of n :

$$\Delta x \Delta p \approx \frac{\pi \hbar n}{\sqrt{3}}.$$

The entropies S_x and S_p as functions of n are not known analytically. Recently, only the asymptotic formula for the position entropy has been found [12]:

$$S_n(x) = \ln(4a) - 1. \quad (4)$$

Equations (3a) and (4) show that the position dispersion and entropy have essentially the same asymptotical behaviour; $\Delta x(n)$ tends to a constant value, while S_x is an n -independent constant. However, the momentum dispersion and entropy have different behaviour which is closely connected with the second-mentioned shortcoming of the uncertainty relation. Since the distance between the main peaks in the momentum probability density increases linearly with n , the momentum dispersion also increases accordingly. On the other hand, the momentum entropy does not essentially depend on this distance and remains practically constant for $n > 4$ (see figure 2(c)). This causes the different behaviour of the uncertainty product $\Delta x \Delta p$ and the sum of entropies $S_x + S_p$.

In order to demonstrate this difference, we have calculated numerically the position and momentum entropy of the potential well as a function of the quantum number for

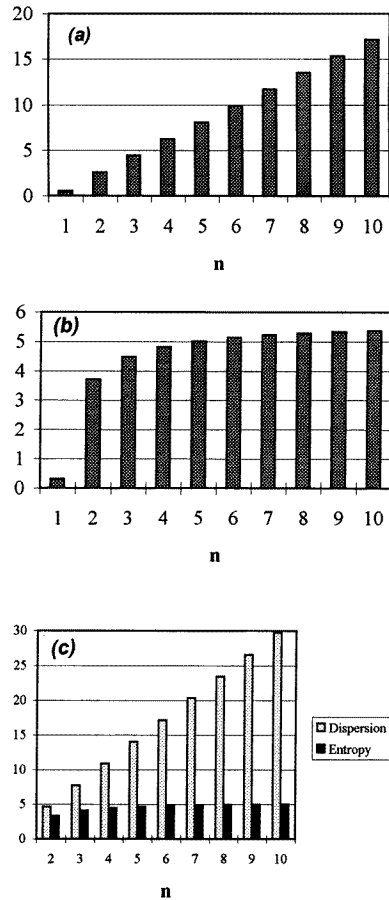


Figure 2. (a) The position–momentum uncertainty product (right-hand side of the standard uncertainty relation), (b) the sum of position and momentum entropies (the right-hand side of entropic uncertainty relation) and (c) dispersion and entropy of momentum as functions of quantum number n .

$n = 1, 2, \dots, 10$. (The calculations were performed using *Mathematica* [5].) To compare the dependence of momentum dispersion and entropy on the quantum number n we plot both quantities in figure 2(c). We see that, whereas the momentum entropy only slightly increases for $n > 2$, the corresponding dispersion increases almost directly proportional to n . In figure 2(a) we plot the uncertainty product of position and momentum (the right-hand side of inequality (2)). This product increases again proportional to the quantum number n . In figure 2(b) we plot the sum of position and momentum entropies (the right-hand side of inequality (1a)). Figure 2 (a and b) shows how both considered quantities depend on the quantum number n . For $n = 1, 2$ and 3 the sum of entropies increases rapidly and then it does only slowly. This indicates that the use of different measures for uncertainty (dispersion or entropy) may lead to different dependences of the uncertainty relations on the quantum number n which is expressively shown for the infinite potential well.

3. Conclusions

From what has been said so far we have the following.

(i) The infinite potential well is a suitable quantum system for demonstrating the difference between the standard and entropic uncertainty relations.

(ii) In figure 1 we see that the probability distribution of position and momentum for $n = 1$ is represented by a one-hump smooth curve whereas for $n > 2$ it is represented by a set of peaks. This causes the uncertainty product of the standard uncertainty relation for $n = 1$ to be small, 0.6 (in unit of \hbar), which is only slightly different from the minimal uncertainty product given by the Heisenberg uncertainty relation ($0.5\hbar$).

(iii) There are two sharp peaks in the momentum probability distribution whose distance increases with quantum number n . Due to this fact the dispersions and entropies differ considerably for $n > 2$. This suggests that the use of entropy as the uncertainty measure corresponds generally more to the demands put upon the measure for the uncertainty than does the dispersion.

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